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HEAVY ION PHYSICS

Analytic determination of the $T-\mu$ phase diagram of the chiral quark model

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Abstract. Using a gap equation for the pion mass a nonperturbative method is given for solving the chiral quark-meson model in the chiral limit at the lowest order in the fermion contributions encountered in a large N_f approximation. The location of the tricritical point is analytically determined. A mean field potential is constructed from which critical exponents can be obtained.

Keywords: large N approximation, tricritical point, finite density
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1. Introduction

Due to the possibility of exploring parts of the phase diagram of strongly interacting matter in current (RHIC) or forthcoming (LHC, GSI) experiments, both numerical (lattice) and analytical investigations have recently received much attention. Theoretically, the phase boundary of QCD depends on the number of quark flavors taken into account and the masses of the quarks [1]. At vanishing chemical potential QCD with two massless quarks undergoes as a function of temperature a second order transition from a hadron to a quark-gluon dominated phase. At zero temperature and large values of the chemical potential the transition from hadronic

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to more exotic phases is of first order. With increasing temperature, the line of the first order transitions ends in a tricritical point (TCP). For non-vanishing masses of the u and d quarks TCP becomes a critical end point (CEP) which separates the first order line from a crossover line defined by the locations of finite maxima of some susceptibilities. The decreasing value of the strange quark has the same effect as increasing the value of the chemical potential, namely it turns the transition into a first order one. Therefore, one can expect that for a physical value of the strange quark mass the CEP in the $\mu - T$ plane will be closer to the temperature axis.

To locate TCP/CEP nonperturbative methods are required. First principle lattice investigations using MC methods based on importance sampling are made useless by the fact that the fermion determinant in the partition function is complex at non-vanishing baryonic chemical potential. To avoid this problem, recently new techniques were developed (for a review see e.g. [2]) to extract, in the $\mu/T \lesssim 1$ region, information for finite μ from simulations at zero chemical potential (multiparameter reweighting of observables, Taylor expansion of the fermion determinant) or at imaginary chemical potential, followed by analytical continuation with or without Taylor expansion. Using the multiparameter reweighting method the CEP was obtained for $n_f = 2 + 1$ dynamical staggered quarks with physical masses in [3]. The location of CEP was estimated in the same range in [4] using an extrapolation based on simulation performed with 3 degenerate quarks.

Analytically, the phase structure of QCD and the location of TCP/CEP can be obtained using the Schwinger-Dyson approach [5] or the strong coupling expansion of the lattice action [6]. Alternatively, one can investigate effective models of QCD [7]. Due to its low energy nature, an effective model might prove not satisfactory in giving the exact location of the endpoint and the shape of the phase boundary, but one can still expect to obtain some insight into the physics near TCP/CEP such as nonequilibrium dynamics. It can tell us what is the soft mode at the CEP, and by predicting specific signatures [8] it may help finding its location. Traces of a possible overlap between the critical regions of TCP and CEP manifesting itself in the $\mu - T$ plane as discussed in [9] could be in principle verified experimentally since it would appear as a gradual change in the value of the critical exponents as the systems pass closer and closer to CEP.

The outline of this paper is as follows. In sec. 2 we introduce an effective quark-meson model and present our method for its solution. In sec. 3 we summarise the main results obtained for the phase diagram and investigate the relation between the effective potential of the theory and the conventional treatment of the TCP in terms of a mean field Landau-type potential. We conclude in sec. 4.

2. The model and the large N_f method of its solution

In the broken symmetry phase characterised by the homogeneous vacuum expectation value Φ the usual shift in the radial direction $\sigma \to \sqrt{N}\Phi + \sigma$ and the requirement of a finite constituent quark mass $m_q = g\Phi$ as $N \to \infty$ defines the following effective

$SU_L(N_f)\times SU_R(N_f)$ symmetric Lagrangian for N mesonic and N_f fermionic fields ($N=N_f^2-1$):

$$L[\sigma,\pi^a,\psi] = -\left[\frac{\lambda}{24}\Phi^4+\frac{1}{2}m^2\Phi^2\right]N-\left[\frac{\lambda}{6}\Phi^3+m^2\Phi\right]\sigma\sqrt{N}$$
$$+\frac{1}{2}\left[(\partial\sigma)^2+(\partial\vec{\pi})^2\right]-\frac{1}{2}m_{\sigma 0}^2\sigma^2-\frac{1}{2}m_{\pi 0}^2\vec{\pi}^2-\frac{\lambda}{6\sqrt{N}}\Phi\sigma\rho^2-\frac{\lambda}{24N}\rho^4$$
$$+\bar{\psi}\left[i\partial^\mu\gamma_\mu-m_q-\frac{g}{\sqrt{N}}\left(\sigma+i\sqrt{2N_f}\gamma_5 T^a\pi^a\right)\right]\psi+\delta L_{ct}[\sigma,\pi^a,\psi], \qquad (1)$$

where $m_{\sigma 0}^2=m^2+\frac{\lambda}{2}\Phi^2$, $m_{\pi 0}^2=m^2+\frac{\lambda}{6}\Phi^2$, $\rho^2=\sigma^2+\vec{\pi}^2$ and $\delta L_{ct}[\sigma,\pi^a,\psi]$ is the counterterm Lagrangian. For fermions the chemical potential is introduced by the replacement: $\partial_t\to\partial_t-i\mu$. One can see that the fermions start contributing at level $\mathcal{O}(1/\sqrt{N})$ in the large N expansion, that is at an intermediate level between LO and NLO of the mesonic sector. Since we want to calculate fermionic corrections to the LO which has O(N) symmetry we disregard the second independent quartic coupling of the $SU(N_f)\times SU(N_f)$ linear σ model which is present for $N_f>2$ and is proportional with the product of two totally symmetric structure constants d.

Our method consists in taking into account perturbatively $N_f=\sqrt{N+1}$ flavors of N_c coloured quarks to $\mathcal{O}(1/\sqrt{N})$ at one loop order both in the EoS

$$V'_{\text{eff}}(\Phi)/\sqrt{N} =: \Phi H(\Phi^2)=\Phi\left[m^2+\frac{\lambda}{6}\Phi^2+\frac{\lambda}{6}T_B(M)-\frac{4g^2N_c}{\sqrt{N}}T_F(m_q)\right]=0, \qquad (2)$$

and in the gap equation $iG_\pi^{-1}(p^2=M^2)=0$ which resums the superdaisy diagrams made of pions with one possible one loop fermion insertion. $T_{B(F)}$ is the bosonic (fermionic) tadpole contribution with mass $M(m_q)$. The inverse pion propagator reads as

$$iG_\pi^{-1}(p)=p^2-m^2-\frac{\lambda}{6}\Phi^2-\frac{\lambda}{6}T_B(M)-B_{1\psi}(p). \qquad (3)$$

The contribution of the fermionic one-loop bubble ("fish" diagram) $B_{1\psi}$ can be related to the fermionic tadpole:

$$B_{1\psi}(p)=-\frac{4g^2N_c}{\sqrt{N}}T_F(m_q)+\frac{2g^2N_c}{\sqrt{N}}p^2 I_F(p,m_q), \qquad (4)$$

where $I_F(p,m_q)$ is the scalar bubble integral with external momentum p evaluated with mass m_q and with Fermi-Dirac distribution, $f_F^\pm(\omega)=1/(\exp(\beta(\omega\mp\mu))+1)$. Substituting relation (4) into (3) and making use of (2) one finds that for $p^2=0$ the inverse propagator vanishes, that is Goldstone's theorem is fulfilled at the minimum of the effective potential i.e. $M=0$. Another way for seeing this is to use the definition of $H(\Phi^2)$ given in (2) in the expression of the gap equation. Then one has

$$H(\Phi^2)=M^2\left(1-\frac{2g^2N_c}{\sqrt{N}}I_F(M,m_q)\right), \qquad (5)$$

which shows that in the spontaneously symmetry broken regime defined by $H(\Phi^2) = 0$, $\Phi \neq 0$ the physical pion mass is always zero.

3. Analytical determination of the TCP

One can choose to study the phase structure by expanding in powers of Φ either the EoS or the effective potential and investigate its behaviour near the transition. In ref. [10] we chose the first possibility and determined the line of continuous transitions by requiring the vanishing of the square bracket of (2) at $\Phi = 0$. The pion mass was fixed at $M = 0$ and only the fermion tadpole was expanded. The line of the 2$^{\text{nd}}$ order phase transitions comes from the condition

$$m_T^2 := m^2 + \frac{g^2\mu^2}{4\pi^2}N_c + \left(\frac{\lambda}{72} + \frac{g^2 N_c}{12}\right)T^2 = 0. \qquad (6)$$

The location of the TCP is determined by requiring in addition to $m_T^2 = 0$ also the vanishing of the coefficient of Φ^2 in the square bracket of (2)

$$\frac{\bar{\lambda}}{6} := \frac{\lambda}{6} + \frac{g^4 N_c}{4\pi^2}\left[\frac{\partial}{\partial n}\Big(Li_n(-e^{-\mu/T}) + Li_n(-e^{-\mu/T})\Big)\Big|_{n=0} - \ln\frac{c_1 T}{M_{0B}}\right] = 0. \qquad (7)$$

Here $\ln(c_1/2) = 1 - \gamma_E + \eta$ and $\eta = \ln(M_{0B}/M_{0F})$ gives a relation between M_{0B} and M_{0F}, the renormalisation scales of the bosonic and fermionic tadpoles. The parameters of the model are fixed at $T = \mu = 0$. Choosing $m_{q0} = m_N/3 \approx 312.67$ and requiring $\Phi_0 = f_\pi/2$ for the solution of the EoS in the case of two flavors ($N_f = 2$) we get $g = 6.72$. $\lambda = 400$ is fixed by requiring a good agreement between the location of the complex pole of the sigma propagator on the 2$^{\text{nd}}$ Riemann sheet and the experimentally favoured mass and width of the sigma particle. Then one can solve the above equations and find the line of 2$^{\text{nd}}$ order phase transitions, the first spinodal line and the location of TCP. This is shown in the l.h.s. of Fig. 1. The other lines in the figure are obtained numerically. In the r.h.s. we see the variation of the TCP and critical temperature with η and M_{0B}. The study of the pole structure of the σ propagator restricts the allowed range of M_{0B} to the right from the arrow, appearing on the horizontal axis of the figure. In this region no low scale imaginary poles appear besides the well known large scale tachyon, whose scale increases with decreasing values of the coupling constant λ. In this region one can arrange very easily $T_c(\mu = 0)$ to occur in the region 150-170 MeV but the TCP stays robustly below 70 MeV. Previous effective model studies [7], and a recent renormalisation group improved investigation [11], give similar values which are very low when compared with the CEP temperature of the QCD obtained in [3] for physical number of quark flavors and masses. This difference could mean that the phase transition is not driven by the light degrees of freedom but by higher excited hadronic states. This possibility was supported by [12] when trying to reproduce lattice results on quark number susceptibilities and pressure with a non-interacting hadron resonance gas model. It was shown in [13] using renormalisation

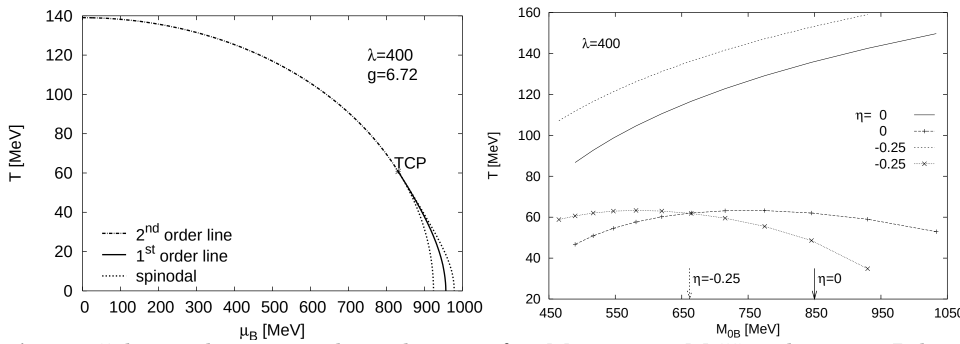

Fig. 1. L.h.s.: The $T-\mu$ phase diagram for $M_{0B}=886$ MeV and $\eta=0$. R.h.s. the dependence of $T_c(\mu=0)$ (upper curves) and of T_{TCP} (lower curves) on M_{0B} for $\eta=0$ and -0.25 (for the definitions see the text).

group techniques that the inclusion of gluon degrees of freedom is important around $T=150$ MeV. On the other hand we should keep in mind that we worked in the limit of an infinitely heavy strange quark, and that lattice studies revealed a very strong dependence on the values of quark masses [14].

We now discuss a second method, in which we keep the Φ-dependence of the mass gap M. The effective potential is obtained by integrating the functional form of EoS away from equilibrium. This analysis reveals an interesting property of the $O(N)$ models in the large N limit: the vanishing of the effective coupling at the critical point [15]. To see this let us first note that $\Phi=0$ becomes a minimum of the potential as $T\to T_c$ and so $M\to 0$. The fourth derivative of the effective potential at this point is $V''''_{\rm eff}(\Phi=0)=4H'(0)$ where $H'(\Phi^2)=\frac{dM^2}{d\Phi^2}\big(1-\frac{2g^2N_c}{\sqrt{N}}I_F(M,m_q)\big)-\frac{2g^2N_c}{\sqrt{N}}M^2\frac{I_F(M,m_q)}{d\Phi^2}$. Since the last term vanishes at T_c one has to study the behaviour of $\frac{dM^2}{d\Phi^2}$ which can be formally expressed from the derivative of EoS with respect to Φ^2:

$$\frac{dM^2}{d\Phi^2}=\frac{\frac{\lambda}{6}-\frac{4g^4N_c}{\sqrt{N}}I_F(0,m_q)+\frac{2g^2N_c}{\sqrt{N}}M^2\frac{dI_F(M,m_q)}{d\Phi^2}}{1-\frac{\lambda}{6}I_B(0,M)-\frac{2g^2N_c}{\sqrt{N}}I_F(M,m_q)}.$$

Using a high temperature expansion $T_B(M^2)\simeq\frac{T^2}{12}-\frac{MT}{4\pi}+\frac{M^2}{8\pi^2}\ln\frac{c_2T}{M_{0B}}$ and one has $\frac{dT_B(M)}{dM^2}\stackrel{M\to 0}{\longrightarrow}-\frac{T}{8\pi M}$, which gives $\frac{dM^2}{d\Phi^2}\big|_{\Phi=0}=\frac{48\pi M}{\lambda}\big(\frac{\lambda}{6}-\frac{4g^4N_c}{\sqrt{N}}\frac{dT_F(m_q)}{dm_q}\big)\stackrel{M\to 0}{\longrightarrow}0.$

This means that along the line of $2^{\rm nd}$ order phase transitions $V''''_{\rm eff}$ vanishes. Still one should be able to provide a link between the effective potential of the model and Landau's theory of the tricritical point in which the location of the TCP is obtained from the condition of a vanishing quartic coupling. In order to achieve this, we first perform a high temperature expansion in $H(\Phi^2)$ given by (2) which yields

$$H(\Phi^2)=m_T^2+\frac{\bar{\lambda}}{6}\Phi^2+\kappa\Phi^4-2UM+\frac{\lambda M^2}{48\pi^2}\ln\frac{c_2T}{M_0},\qquad(8)$$

where $U = \frac{\lambda T}{48\pi}$ and $\kappa = \frac{g^6 N_c}{16\pi^2\sqrt{N}} \frac{1}{T^2} \frac{\partial}{\partial n} \left[\mathrm{Li}_n(-e^{\mu/T}) + \mathrm{Li}_n(-e^{-\mu/T}) \right] \big|_{n=-2}$. In the following discussion for the sake of simplicity we omit in (5) the one-loop fermion contribution $I_F(M, m_q)$. We then rewrite the resulting $H(\Phi^2) = M^2$ relation as an equation which determines the Φ^2 dependence of M

$$WM^2 + 2UM - \left(m_T^2 + \frac{\bar{\lambda}}{6}\Phi^2 + \kappa\Phi^4 \right) = 0. \tag{9}$$

where $W = 1 - \frac{\lambda^2}{48\pi^2} \ln \frac{c_2 T}{M_0}$ and $\ln \frac{c_2}{4\pi} = \frac{1}{2} - \gamma$. The solution of (9) approximates $H(\Phi^2)$, so after substituting it into (2) it can be integrated to obtain an approximation to the effective potential. Subsequent expansion in powers of Φ^2 yields

$$V_{\mathrm{approx}}(\Phi)/\sqrt{N} = \frac{m_T^2}{4U^2 W} \left[\frac{m_T^2}{2}\Phi^2 + \frac{\bar{\lambda}}{12}\Phi^4 + \left(\frac{W\bar{\lambda}^2}{216 m_T^2} + \frac{\kappa}{3} \right) \Phi^6 \right]. \tag{10}$$

A few remarks are in order here. *First of all* whether this potential has the right shape, and whether this approximation has the correct physical meaning depends on the actual values of the parameters. One can easily verify that for the parameters used in drawing Fig. 1, $W < 0$ in the temperature range of the 2^{nd} order phase transition. For increasing values of μ/T, κ changes sign at 1.91 from negative to positive values. Since around TCP κ has to be positive, this sign change restricts the location of the TCP to large values of μ: it must be in the region $\mu > 2T$. *Second*, in the broken symmetry phase $m_T^2 < 0$ and it vanishes at T_c meaning that the coefficient of the quartic term in Φ vanishes also. Around T_c from (6) one gets $m_T^2 \sim T - T_c$. Therefore the scaling of the effective mass $m_{\mathrm{eff}} = \left| \frac{m_T^2}{2UW} \right| \sim |T - T_c|$ gives the correct leading order critical exponent $\nu = 1$ for the O(N) model at large N. *Third*, to the expression in the square bracket in (10) the usual Landau type analysis applies. The location of the TCP is determined by the same conditions as in the analysis based on EoS, namely (6) and (7). To obtain the scaling exponent of the order parameter on the second order line we need only the first two terms in (10). The minimum condition gives $\Phi^2 = -\frac{3m_T^2}{\bar{\lambda}} \Rightarrow \Phi \sim (T - T_c)^\beta$, $\beta = 1/2$. At TCP one can set $\bar{\lambda} = 0$ in (10) and keep the sixth order term. The minimum condition gives $\Phi^4 = -\frac{m_T^2}{2\kappa} \Rightarrow \Phi \sim (T - T_c)^\beta$, $\beta = 1/4$. We verified numerically using the exact EoS that the value of β on the second order line and at the TCP agrees within the numerical accuracy with their mean field estimates given above.

Refinement of the approximation is needed if one wants to enter the metastable region between the spinodals where $m_T^2 > 0$ and $\bar{\lambda} < 0$, but we think that nevertheless the presented analysis captures essential features of the phenomenon.

4. Conclusions

In the $1/\sqrt{N}$ order of the large N approximation to the solution of the chiral quark-meson model we presented in the chiral limit an analytical determination of the phase diagram and of the location of the tricritical point. We have shown that with a reasonable approximation which works fine at high temperature, the effective potential of the theory can be related to mean field Landau type potential, from which the critical exponents can be determined.

In order to assess to what extent a description of the phase diagram of QCD is possible using a low-energy effective model, one has both to improve our method of solving the model and to get as close as possible to the physical case of $2+1$ flavors. The first issue necessitates taking into account the full momentum dependence of the self-consistent pion propagator, the implementation of a new resummation method and solving the Dyson-Schwinger equation for the fermions. The latter problem requires the extension of our method to the $SU(3)_L \times SU(3)_R$ case.

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